

STOILLOW'S THEOREM REVISITED

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ABSTRACT. Stoilow's theorem from 1928 states that a continuous, light, and open mapping between surfaces is a discrete map with a discrete branch set. This result implies that such mappings between orientable surfaces are locally modelled by power mappings $z \mapsto z^k$ and admit a holomorphic factorization.

The purpose of this expository article is to give a proof of this classical theorem having the readers interested in discrete and open mappings in mind.

1. INTRODUCTION

Stoilow's classical theorem in [15] states that *a light and open continuous mapping between surfaces is a discrete map which has a discrete branch set*. In what follows, we call this theorem as *Stoilow's discreteness theorem*.

Recall that a continuous mapping $f: X \rightarrow Y$ between topological spaces is *light* if the pre-image $f^{-1}(y)$ of each point $y \in Y$ is totally disconnected, and *discrete* if $f^{-1}(y)$ is a discrete set. A continuous mapping is *open* if the image of each open set is an open set. The branch set B_f of a continuous mapping $f: X \rightarrow Y$ is the set of points $x \in X$ at which f fails to be a local homeomorphism.

In [15] Stoilow shows that these mappings are locally modelled by power maps $z \mapsto z^k$ [15, p.372]. This local description indicates a deep connection between light and open mappings and holomorphic mappings between surfaces. This connection was coined by Stoilow [16, p.120] and Whyburn [18, Theorem X.5.1, p.198] and [19, p.103]: *For a light and open continuous mapping $f: S \rightarrow S'$ between orientable Riemann surfaces there exists a Riemann surface \tilde{S} and a homeomorphism $h: S \rightarrow \tilde{S}$ for which $f \circ h^{-1}: \tilde{S} \rightarrow S'$ is a holomorphic map*; the Riemann surface \tilde{S} in this statement is naturally the Riemann surface associated to the map f . The first edition of [19], published in 1956, does not give this result a specific name, but already in the second edition from 1964 the result is referred as *Stoilow's theorem*.

In this expository article we discuss the proof of Stoilow's discreteness theorem having readers interested in discrete and open mappings, such as quasiregular mappings (see e.g. [14]) or Thurston maps (see e.g. [4]), in mind. For this reason we separate Stoilow's theorem into two parts: the discreteness of the map and the discreteness of the branch set.

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Theorem 1.1 (Stoilow (1928)). *Let Ω be a domain in \mathbb{C} and $f: \Omega \rightarrow \mathbb{C}$ a light, open, and continuous mapping. Then f is a discrete map.*

Theorem 1.2 (Stoilow (1928)). *Let Ω be a domain in \mathbb{C} and $f: \Omega \rightarrow \mathbb{C}$ a discrete, open, and continuous mapping. Then B_f is a discrete set.*

It is interesting to notice that both results stem from path-lifting arguments. Indeed, after path-lifting results are established, standard applications of the Jordan curve theorem together yield Theorems 1.1 and 1.2 almost as corollaries. For discrete and open mappings, we may use Rickman's path-lifting theorem [13] and for light and open mappings method of Floyd [6]. Floyd's method suffices for all our purposes and we recall it in Section 3; see [10] for a more detailed discussion on path-lifting methods.

Having Theorems 1.1 and 1.2 at our disposal, it is a straightforward covering space argument to show that light open mappings between surfaces are locally modelled by power maps.

Theorem 1.3 (Stoilow (1928)). *Let $f: \Sigma \rightarrow \Sigma'$ a continuous, light, and open map between surfaces. For each $z \in \Sigma$, there exists $k \in \mathbb{N}$, a neighborhood U of z , and homeomorphisms $\psi: U \rightarrow \mathbb{D}$ and $\phi: fU \rightarrow \mathbb{D}$ for which the diagram*

$$\begin{array}{ccc} U & \xrightarrow{f|_U} & fU \\ \psi \downarrow & & \downarrow \phi \\ \mathbb{D} & \xrightarrow{z \mapsto z^k} & \mathbb{D} \end{array}$$

commutes.

This local version of Stoilow's discreteness theorem yields the global factorization theorem. Indeed, by Theorem 1.3, a light and open mapping $f: \Sigma \rightarrow S'$ induces a conformal structure on Σ making it a Riemann surface \tilde{S} ; we refer to [7, Section 1.1] or [4, Lemma A.10] for a short proof. Thus we may consider f as a holomorphic map $f: \tilde{S} \rightarrow S'$. If the surface Σ a priori carries a conformal structure, and we consider Σ as a Riemann surface S , we may take the homeomorphism h in the factorization to be the identity homeomorphism $\Sigma \rightarrow \Sigma$.

Theorem (Stoilow's factorization theorem (1938)). *Let $f: S \rightarrow S'$ be a light, open, and continuous mapping between Riemann surfaces. Then there exists a Riemann surface \tilde{S} and a homeomorphism $h: S \rightarrow \tilde{S}$ such that $f \circ h^{-1}: \tilde{S} \rightarrow S'$ is a holomorphic map.*

Recall that an orientable topological surface carries a conformal structure. Indeed, by a classical theorem of Radó, every 2-manifold can be triangulated (see e.g. [11, Theorem 8.3, p. 60] or [1, Section II.8, p.105]) and every triangulated orientable surface has a conformal structure (see e.g. [1, II.2.5E, Theorem, p.127] or [3, Section 2.2, pp. 9-11]). In this way we recover the interpretation that *a light open mapping between orientable surfaces is a holomorphic map between Riemann surfaces.*

As authors' interest to these theorems of Stoilow stems from their role in the theory of quasiregular mappings, we finish this introduction with a

related remark. In the quasiconformal literature Stoilow's theorem typically refers to the result that *each quasiregular mapping $S \rightarrow S'$ between Riemann surfaces factors to a holomorphic map $S \rightarrow S'$ and a quasiconformal homeomorphism $S \rightarrow S$* . Proof of this result is analytic in its nature and based on the Beltrami equation. We refer to Astala, Iwaniec, and Martin [2, Theorem 5.5.1] for a detailed discussion.

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2. PRELIMINARIES

A totally disconnected closed set in the plane has topological dimension zero. We recall this fact in the following form; see [9, Section II.2, p.14] for a proof.

Fact 2.1. *Let C be a closed and totally disconnected set in \mathbb{C} . Then every point $x \in C$ has a neighbourhood basis consisting of neighbourhoods U satisfying $C \cap \partial U = \emptyset$.*

Let Ω be a planar domain and $f: \Omega \rightarrow \mathbb{C}$ be a continuous, light, and open mapping. A domain $U \subset \Omega$ is a *normal domain* if U is compactly contained in Ω and $\partial fU = f\partial U$. A normal domain U is a *normal domain of x* if it is also a neighbourhood of x . We call U a *normal neighbourhood of x* , if $\overline{U} \cap f^{-1}\{f(x)\} = \{x\}$. Note that for a normal domain U , the restriction $f|_U: U \rightarrow fU$ is a proper map, i.e. the pre-image $f^{-1}K$ of a compact set $K \subset fU$ is compact.

For each $x \in \Omega$ and $r > 0$, we denote $U(x, f, r)$ the component of $f^{-1}B(f(x), r)$ containing the point x . The following lemma shows that domains $U(x, f, r)$ are normal domains of x for all $r > 0$ small enough; cf. [17, Lemma 5.1].

Lemma 2.2. *Let Ω be a planar domain, $f: \Omega \rightarrow \mathbb{C}$, a continuous, light, and open mapping, and let $x \in \Omega$. Then there exists a radius $r_x > 0$ such that for all $r \leq r_x$ the domain $U(x, f, r)$ is a precompact normal domain of x and $fU(x, f, r) = B(f(x), r)$.*

Proof. Let $x \in \Omega$. By Fact 2.1 there exists a precompact neighbourhood V of x for which $\partial V \cap f^{-1}\{f(x)\} = \emptyset$. Since ∂V is compact and does not contain pre-images of x , we may fix $r > 0$ for which $B(f(x), r) \cap f\partial V = \emptyset$.

We set $U := U(x, f, r)$ and observe first that U is precompact. Indeed,

$$f(U \cap \partial V) \subset fU \cap f\partial V \subset B(f(x), r) \cap f\partial V = \emptyset.$$

Thus $U \cap \partial V = \emptyset$ and $U \subset V$ by connectedness, so $\overline{U} \subset \overline{V} \subset \Omega$ is compact.

We claim next that $\partial fU = f\partial U$. Since f is open, $\partial fU \subset f\partial U$. On the other hand, let $z \in \overline{U} \cap f^{-1}B(f(x), r)$, and let W be the component of $V \cap f^{-1}B(f(x), r)$ containing z . Then W is a neighbourhood of z which intersects U . Since U is a component of $f^{-1}B(f(x), r)$, we conclude that $W = U$ and $z \in U$. Hence $f\partial U \subset \partial fU$.

This shows that U is a normal domain and that fU is both open and closed in $B(f(x), r)$. Thus $fU = B(f(x), r)$. \square

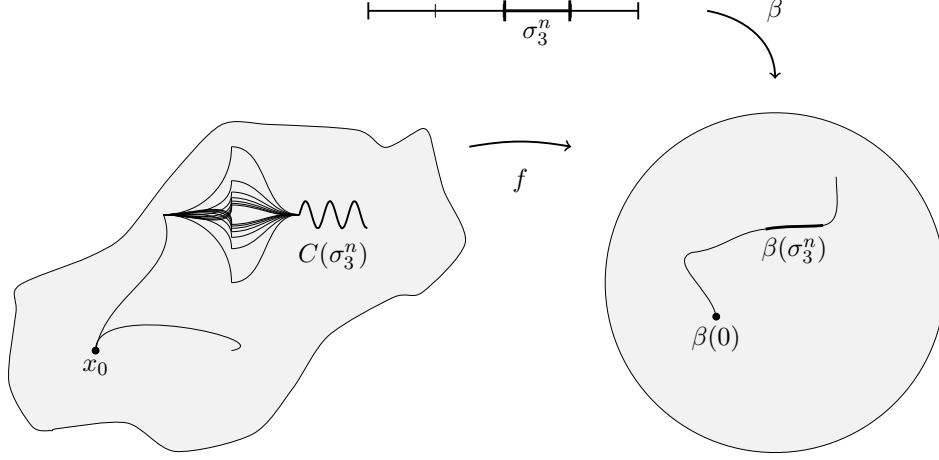


FIGURE 1. Construction of the lift of a path in Theorem 3.1.

3. PATH LIFTING AFTER FLOYD AND STOILOW

We now turn to path-lifting, which is one of the main tools in the forthcoming proofs. The following theorem is, essentially, due to Stoilow [15, pp. 354-358] and its idea was generalized by Floyd [6] to the setting compact metric spaces. We include here a version of Floyd's proof in the planar setting for the reader's convenience.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a planar domain and $f: \Omega \rightarrow \mathbb{C}$ a continuous, light, and open mapping. Let U be a normal domain compactly contained in Ω . Then for any path $\beta: [0, 1] \rightarrow fU$ and any point $x_0 \in U \cap f^{-1}\{\beta(0)\}$ there exists a path $\alpha: [0, 1] \rightarrow U$ satisfying $\alpha(0) = x_0$ and $f \circ \alpha = \beta$.*

For the proof of Theorem 3.1 we need the following elementary lemma; cf. [18, pp. 131, 148].

Lemma 3.2. *Let $\Omega \subset \mathbb{C}$ be a planar domain and $f: \Omega \rightarrow \mathbb{C}$ a continuous, light, and open mapping. Suppose U is a normal domain compactly contained in Ω . Then for any $\varepsilon > 0$ there exists a constant $\delta > 0$ having the property that, for any continuum $C \subset fU$ satisfying $\text{diam}(C) < \delta$, the components of $U \cap f^{-1}C$ have diameter strictly less than ε .*

Proof. Suppose there exists $\varepsilon_0 > 0$ having the property that, for each $n \in \mathbb{N}$, there exists a continuum $C_n \subset fU$ having diameter at most $1/n$ and a component C'_n of $U \cap f^{-1}C_n$ having diameter at least ε_0 .

Since both \overline{U} and $f\overline{U}$ are compact we may, by passing to subsequences, assume that the sequences (C_n) and (C'_n) converge in the Hausdorff distance to a continuum $C \subset f\overline{U}$ and to a point $C' \in \overline{U}$, respectively. Then $fC = C'$. Since f is light, this is a contradiction. Thus the claim holds true. \square

Proof of Theorem 3.1. By the uniform continuity of β and Lemma 3.2, there exists an increasing sequence of integers (m_n) such that, for any interval $J \subset [0, 1]$ satisfying $\text{diam}(J) \leq 2^{-m_n}$, each component of $U \cap f^{-1}(\beta J)$ has diameter of at most 2^{-n} .

For each $n \in \mathbb{N}$ we denote Σ^n the barycentric division of $[0, 1]$ into intervals $\sigma_k^n = [(k-1)2^{-m_n}, k2^{-m_n}]$ for $k = 1, \dots, 2^{m_n}$. We say that intervals σ and σ' in Σ^n are *adjacent* if σ and σ' meet at an end point, that is, $\sigma \neq \sigma'$ and $\sigma \cap \sigma' \neq \emptyset$.

We construct next, for each $n \in \mathbb{N}$, a collection of continua $\{C(\sigma)\}_{\sigma \in \Sigma^n}$ with the following properties:

- (1) $C(\sigma)$ is a component of $f^{-1}\beta(\sigma)$ having $\text{diam}(C(\sigma)) < 2^{-n}$,
- (2) if $0 \in \sigma$, then $x_0 \in C(\sigma)$, and
- (3) $C(\sigma) \cap C(\sigma') \neq \emptyset$ for adjacent intervals $\sigma, \sigma' \in \Sigma^n$;

see Figure 1.

Let $n \in \mathbb{N}$. We define the continua $C(\sigma_1^n), \dots, C(\sigma_r^n)$, where $r = 2^{m_n}$, inductively as follows. Let $C(\sigma_1^n)$ be a component of $U \cap f^{-1}\beta(\sigma_1^n)$ containing x_0 . Suppose that we have defined continua $C(\sigma_1^n), \dots, C(\sigma_k^n)$ for some $1 \leq k < r$, and let $C(\sigma_{k+1}^n)$ be a component of $U \cap f^{-1}\beta(\sigma_{k+1}^n)$ which intersects $C(\sigma_k^n)$. Such a component exists, since for any compact set $K \subset fU$, each component of $U \cap f^{-1}K$ maps surjectively onto K . The collection $\{C(\sigma)\}_{\sigma \in \Sigma^n}$ of continua now satisfies the conditions (1)-(3).

Let $n \in \mathbb{N}$ and $\sigma \in \Sigma^n$. For each $m \geq n$, let $C^{n,m}(\sigma)$ be the union

$$\bigcup \{C(\tau) \mid \tau \in \Sigma^m, \tau \subset \sigma\}.$$

It is straightforward to verify that $C^{n,m}(\sigma)$ is connected and $f(C^{n,m}(\sigma)) = \beta(\sigma)$. Furthermore, since $\beta([0, 1]) \subset fU$ and U is a normal domain, there exists a uniform positive lower bound for the distance between $C^{n,m}(\sigma)$ and ∂U . Thus, for each $n \in \mathbb{N}$ and $\sigma \in \Sigma^n$, there exists a subsequence of $(C^{n,m}(\sigma))_{m=n}^\infty$ converging to a continuum $C^n(\sigma) \subset U$ in the Hausdorff distance. By a diagonal argument, we may assume that

- (i) for each $n \in \mathbb{N}$ and $\sigma \in \Sigma^n$, the sequence $(C^{n,m}(\sigma))_{m=n}^\infty$ converges to $C^n(\sigma)$, and
- (ii) for all $n \leq k \leq m$ and intervals $\sigma \in \Sigma^n$ and $\sigma' \in \Sigma^k$ satisfying $\sigma' \subset \sigma$, we have $C^{k,m}(\sigma') \subset C^{n,m}(\sigma)$.

Note that $\text{diam}(C^n(\sigma)) \leq 2^{-n}$, and $C^n(\sigma) \cap C^n(\sigma') \neq \emptyset$ for adjacent intervals $\sigma, \sigma' \in \Sigma^n$.

After these preliminaries we are now ready to define a path $\alpha: [0, 1] \rightarrow U$ as follows. Let $t \in [0, 1]$. For each $n \in \mathbb{N}$, let σ_n and σ'_n be the intervals in Σ^n containing t ; possibly with $\sigma_n = \sigma'_n$. Then the sequence $(C^n(\sigma_n) \cup C^n(\sigma'_n))_n$ is a decreasing sequence of continua whose diameters tend to zero. Thus we may define $\alpha(t)$ by

$$\{\alpha(t)\} = \bigcap_n (C^n(\sigma_n) \cup C^n(\sigma'_n)).$$

For each pair σ and σ' of adjacent intervals in Σ^n we have

$$\alpha(\sigma \cup \sigma') \subset C^n(\sigma) \cup C^n(\sigma'),$$

where $C^n(\sigma) \cup C^n(\sigma')$ has diameter of at most $2 \cdot 2^{-n}$. Thus α is continuous. By construction, $f \circ \alpha = \beta$ and $\alpha(0) = x_0$. This concludes the proof. \square

We end this section with uniqueness for lifts into simply connected planar domains. Note that this claim clearly fails for mappings between more general surfaces, even between spheres, and in higher dimensions.

Proposition 3.3. *Let $\Omega \subset \mathbb{C}$ be a simply connected planar domain, $f: \Omega \rightarrow \mathbb{C}$ a continuous open map, and let $\beta_1, \beta_2: [0, 1] \rightarrow \Omega$ be lifts of the same arc $\alpha: [0, 1] \rightarrow f\Omega$ for which $\beta_1(0) = \beta_2(0)$ and $\beta_1(1) = \beta_2(1)$. Then $\beta_1 = \beta_2$.*

The proof of Proposition 3.3 is an almost immediate consequence of the following version of the Jordan curve theorem; for a proof we refer to [11, Section 4] or [12, Theorem 1.10, p.33].

Theorem (The Jordan curve theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected planar domain and let $c: \mathbb{S}^1 \rightarrow \Omega$ be an injective continuous map. Then $\Omega \setminus |c|$ consists of two domains, exactly one of which has a compact closure in Ω . Both of these domains have the image of the curve c as their boundary.*

Proof of Proposition 3.3. Suppose $\beta_1 \neq \beta_2$. Then there exists $t_0 \in [0, 1]$ for which $\beta_1(t_0) \neq \beta_2(t_0)$. Let

$$a = \sup\{t \in [0, t_0] \mid \beta_1(t) = \beta_2(t)\} \text{ and } b = \inf\{t \in [t_0, 1] \mid \beta_1(t) = \beta_2(t)\}.$$

Then there exists a closed loop $c: \mathbb{S}^1 \rightarrow \Omega$ for which $|c| = |\beta_1|_{[a,b]} \cup |\beta_2|_{[a,b]}$, and so $|f \circ c| = \alpha[a, b]$. By the Jordan Curve Theorem, one of the components of $\Omega \setminus |c|$, say U , is a precompact domain of Ω with $\partial U = |c|$.

Thus fU is a precompact domain in \mathbb{C} and $\partial fU \subset |\alpha|$. This is a contradiction since no such precompact domain exists since α is an arc. The claim follows. \square

4. PROOF OF THEOREM 1.1

We use now path lifting and the Jordan Curve Theorem to prove Theorem 1.1, that is, to show that a continuous, light, and open planar mapping is discrete. We follow here that idea of Stoilow that the discreteness follows from the finiteness of lifts of rays. Stoilow calls the following proposition, together with the existence of lifts, *Théorème Fondamental* ([15, p. 361]).

Proposition 4.1. *Let Ω be a plane domain and $f: \Omega \rightarrow \mathbb{C}$ a light, open and continuous map. Let $x \in \Omega$ and let $r > 0$ be so small that $U := U(x, f, r)$ is a normal domain contained in a simply connected neighbourhood in Ω . Suppose $\beta: [0, 1] \rightarrow B(f(x), r)$ is a ray $t \mapsto tz_0 + f(x)$, where $x_0 \in \mathbb{S}^1$. Then there exists at most finitely many lifts of β in U .*

Having this proposition at our disposal, Theorem 1.1 follows immediately.

Proof of Theorem 1.1 assuming Proposition 4.1. Let $x_0 \in \Omega$ be a point and fix a normal domain U_0 of x_0 contained in some simply connected neighbourhood in Ω . By Proposition 4.1, a ray in fU_0 starting from $f(x_0)$ has only finitely many lifts in U_0 . Thus the set $U_0 \cap f^{-1}\{f(x_0)\}$ is finite. \square

Proof of Proposition 4.1. Suppose a ray $\beta: [0, 1] \rightarrow fU$ has infinitely many mutually distinct lifts $\tilde{\beta}_n: [0, 1] \rightarrow U$, $n \in \mathbb{N}$. By passing to a subsequence if necessary, we may assume that $\tilde{\beta}_n(0) \rightarrow x_0 \in U$ and $\tilde{\beta}_n(1) \rightarrow y_0 \in U$ as $n \rightarrow \infty$.

There are two non-exclusive possibilities for the sequence $(\tilde{\beta}_n)$:

- (a) there exists a subsequence where all paths are mutually disjoint, or
- (b) there exists a subsequence where any two lifts intersect.

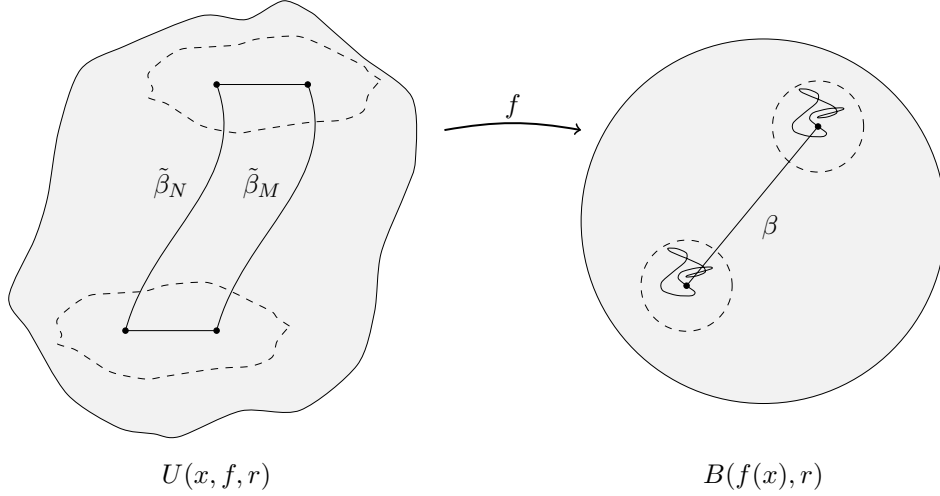


FIGURE 2. Constructing a “rectangular Jordan curve” in the proof of Proposition 4.1.

Indeed, for each n there exists either infinitely many lifts $\tilde{\beta}_m, m \geq n$, intersecting $\tilde{\beta}_n$ or infinitely many lifts $\tilde{\beta}_m$ not intersecting $\tilde{\beta}_n$. By a diagonal argument we may fix a subsequence $(\tilde{\beta}_{n_k})$ of $(\tilde{\beta}_n)$ such that for each $k \in \mathbb{N}$, either $|\tilde{\beta}_{n_k}| \cap |\tilde{\beta}_{n_m}| = \emptyset$ for all $m \geq k$ or $|\tilde{\beta}_{n_k}| \cap |\tilde{\beta}_{n_m}| \neq \emptyset$ for all $m \geq k$. Since there exists infinitely many indices k for which one of these two possibilities hold, we may pass to a subsequence of $(\tilde{\beta}_{n_k})$ to receive a subsequence of type (a) or (b), respectively. For the rest of the proof we may assume that the original sequence $(\tilde{\beta}_n)$ itself satisfies the condition in (a) or (b).¹

In case (a) we fix a radius $s > 0$ for which $V := U(x_0, f, s)$ and $V' := U(y_0, f, s)$ are mutually disjoint normal domains of x_0 and y_0 , respectively. Let $M, N \in \mathbb{N}$ be indices for which $\tilde{\beta}_M(0), \tilde{\beta}_N(0) \in V$ and $\tilde{\beta}_M(1), \tilde{\beta}_N(1) \in V'$. Then there exists a Jordan curve $c: \mathbb{S}^1 \rightarrow U$ such that

$$|c| \subset V \cup V' \cup |\tilde{\beta}_M| \cup |\tilde{\beta}_N|$$

and $|c| \not\subset V \cup V'$; see Figure 2. The image $|f \circ c|$ is a continuum contained in the ray $|\beta|$ and two mutually disjoint disks, fV and fV' , located at the endpoints of β . By the Jordan Curve Theorem, the curve c bounds a precompact domain W in U , having boundary contained in $|c|$. Since fW is not precompact, this is a contradiction.

In case (b), let $\gamma: [0, 1] \rightarrow B(f(y_0), r)$ be a ray for which $\gamma(0) = f(y_0)$ and $|\gamma| \cap |\beta| = f(y_0)$; see Figure 3. For each $n \in \mathbb{N}$, let $\tilde{\gamma}_n$ be a lift of γ starting from $\tilde{\beta}_n(1)$. The paths $\tilde{\gamma}_n$ are mutually disjoint, since otherwise a restriction of the composition of β and γ contradicts Proposition 3.3. Now the argument of case (a) applies to the sequence $(\tilde{\gamma}_n)$. \square

¹This argument does not rely on any special properties of the lifts or the geometry of the plane, only on combinatorial data on intersections – this is the infinite Ramsey theorem for two colors; see e.g. [8, Theorem 5, p. 16].

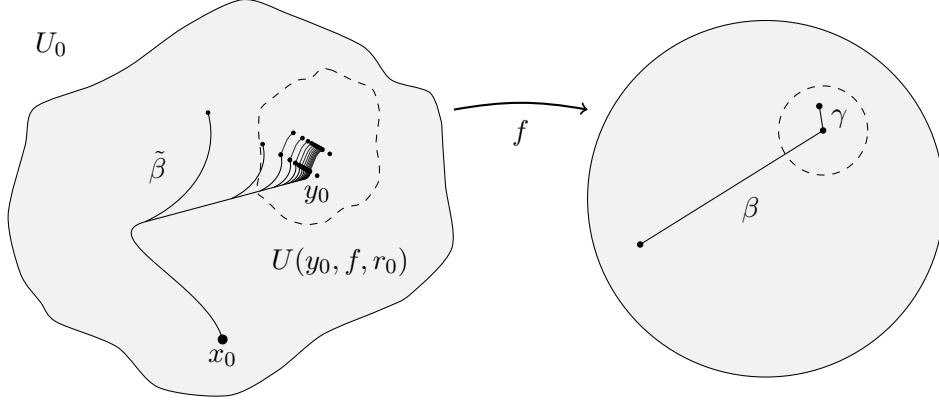


FIGURE 3. Path having infinitely many lifts with a joint starting point in the proof of Proposition 4.1.

5. PROOFS OF THEOREMS 1.2 AND 1.3

Theorem 1.2 is a direct consequence of the following proposition.

Proposition 5.1. *Let Ω be a planar domain and let $f: \Omega \rightarrow \mathbb{C}$ be a continuous, open, and discrete mapping. Let $x_0 \in \Omega$ and let $r > 0$ be so small that $U_0 := U(x_0, f, r)$ is a normal neighbourhood of x_0 contained in a simply connected domain in Ω . Then $U_0 \cap B_f \subset \{x_0\}$.*

Note that since f is a priori both continuous and open, the branch set B_f of f is actually the set of points at which f is not locally injective.

Proof of Proposition 5.1. Suppose there exists $b \in (U_0 \cap B_f) \setminus \{x_0\}$ and let $U \subset U_0$ be a normal neighborhood of b . Since $b \in B_f$, the mapping f is not locally injective at b . Thus we may fix $y_0 \in fU$ for which $\#(U \cap f^{-1}\{y_0\}) \geq 2$.

Let $\alpha: [0, 1] \rightarrow fU$ and $\beta: [0, 1] \rightarrow fU_0 \setminus \alpha(0, 1]$ be arcs satisfying $\alpha(0) = \beta(0) = y_0$, $\alpha(1) = f(b)$, and $\beta(1) = f(x_0)$. Let also $z_1, z_2 \in U \cap f^{-1}\{y_0\}$, $z_1 \neq z_2$. By Theorem 3.1, there exists, for $i = 1, 2$, lifts $\tilde{\alpha}_i: [0, 1] \rightarrow U$ and $\tilde{\beta}_i: [0, 1] \rightarrow U_0$ of α and β , respectively, satisfying $\tilde{\alpha}_i(0) = \tilde{\beta}_i(0) = z_i$ for $i = 1, 2$. Since U_0 and U are normal neighbourhoods of x_0 and b , respectively, we have $\tilde{\alpha}_i(1) = b$, $\tilde{\beta}_i(1) = x_0$ for $i = 1, 2$.

For $i = 1, 2$, let $\tilde{\gamma}_i: [0, 1] \rightarrow U$ be the path

$$t \mapsto \begin{cases} \tilde{\alpha}_i(2t), & t \in [0, 1/2] \\ \tilde{\beta}_i(2t - 1), & t \in [1/2, 1]. \end{cases}$$

Since $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$, $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$, and U is contained in a simply connected domain, we have by Proposition 3.3 that $\tilde{\gamma}_1 = \tilde{\gamma}_2$. This is a contradiction and the claim follows. \square

We finish with a simple proof of Theorem 1.3 based on Theorem 1.2 and a covering argument.

Proof of Theorem 1.3. Let $z \in \Sigma$. Since the property is local, we may assume that Σ and Σ' are planar domains.

Let $r > 0$ be so small that $U(z, f, r)$ is a normal neighborhood of z contained in some simply connected domain in Σ . Denote $U' = U(z, f, r) \setminus \{z\}$, $B' = B(f(z), r) \setminus \{f(z)\}$, and $\mathbb{D}' = \mathbb{D} \setminus \{0\}$. Since $f|_{U'}: U' \rightarrow B'$ is a local homeomorphism and a proper map, it is a covering map. Furthermore, since f is discrete,

$$U' \cap f^{-1}\{y\} \subset \overline{U} \cap f^{-1}\{y\}$$

is finite for any $y \in fU$. As a finite cover of \mathbb{D}' , U' is a topological punctured disk and we conclude that U is a topological disk.

Let $h_1: U \rightarrow \mathbb{D}$ and $h_2: B(f(z), r) \rightarrow \mathbb{D}$ be homeomorphisms with $h_1(z) = 0$ and $h_2(f(z)) = 0$. Denote

$$g := h_2 \circ f \circ (h_1|_{U'})^{-1}: \mathbb{D}' \rightarrow \mathbb{D}'.$$

Since $f|_{U'}: U' \rightarrow B'$ is a covering map, so is g . Thus the induced map $g_*: \pi_1(\mathbb{D}') \rightarrow \pi_1(\mathbb{D}')$ is of the form $m \mapsto km$ for some $k \in \mathbb{Z} \setminus \{0\}$. Denote $\zeta_k := z \mapsto z^k$, and let $h': \mathbb{D}' \rightarrow \mathbb{D}'$ be the lift of $g: \mathbb{D}' \rightarrow \mathbb{D}'$ under the covering map ζ_k . Then $g = \zeta_k \circ h'$ and h' is a homeomorphism, since it is an injective covering map. The homeomorphism h' extends to a homeomorphism $h: \mathbb{D} \rightarrow \mathbb{D}$ by the continuity of ζ_k and hence $f|_U = h_2 \circ \zeta_k \circ h \circ h_1 =: \phi^{-1} \circ \zeta_k \circ \psi$. \square

Remark 5.2. We find it interesting that the argument of this corollary together with the Černavskii-Väisälä theorem (see [5] and [17]) on the branch set of discrete and open mappings gives another proof for the discreteness of the branch set in the case of discrete and open mappings between surfaces.

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